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An MHD Channel Flow with Temperature Dependent Electrical Conductivity

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Magnetohydrodynamic channel flows have received considerable attention of late, and several solutions using approximate methods have been obtained.[1–4] The motivation given for some of these solutions has been the current interest in MHD power generators. However the models used have been appropriate only for certain liquid metal flows since the assumptions made included 1) an incompressible fluid with constant properties and 2) zero Hall parameter. These assumptions will not be valid in a useful gaseous MHD generator, and it is necessary to consider the effect of relaxing these restrictions before extrapolating these results to channel flows of an electrically conducting gas. The effect of the Hall parameter has been considered by Tani,[5] and here we will consider the effect of a temperature-dependent electrical conductivity.

Hale and Kerrebrock[6] have considered a compressible boundary layer with temperature-dependent fluid properties in an MHD accelerator and have shown that both velocity and temperature overshoots (velocities and temperatures greater than the freestream values) exist in the boundary layer on the insulating wall. It is the purpose of this work to demonstrate by analyzing a particularly simple model how these peculiar velocity distributions can occur even when the temperature distribution is approximately parabolic, and the effect of the MHD terms in the energy equation is small. This is in contrast to the results of Hale and Kerrebrock and an earlier note by Lykoudis[7] where the velocity overshoots are a direct result of the temperature (or conductivity) overshoots inside the boundary layer. A channel flow is chosen as the model since at high Hartmann numbers this illustrates several features of a boundary layer in that viscous forces are only important close to the channel walls.

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Problem Formulation

Consider the steady fully-developed flow of an electrically conducting gas through a channel of rectangular cross section as shown in Fig. 1. It is assumed that $a \gg 2h$ so that the flow is two-dimensional; that the specific heat c_p, viscosity μ, thermal conductivity k, Prandtl number Pr, and density ρ of the gas are constant; that E and B, the imposed electric and magnetic fields are constant; and that the magnetic Reynolds number is small. For these assumptions the x momentum equation reduces to

$$\mu(d^2u/dy^2) - dp/dx - jB = 0 \tag{1}$$

where the current density j which flows in the z direction is given by

$$j = \sigma(uB + E) \tag{2}$$

where σ is the electrical conductivity, which is a function of temperature. The energy equation reduces to

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{j^2}{\sigma} + u\frac{dp}{dx} + k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{du}{dy}\right)^2 \tag{3}$$

The dimensionless temperature $\theta = (T - T_w)/(T_c - T_w)$ is now introduced and is assumed to be a function of y only; subscripts w and c refer to wall and centerline values, respectively. The electrical conductivity is assumed to be given by the relation

$$\sigma/\sigma_c = \theta^\omega \tag{4}$$

where ω is a positive exponent. Equations (1) and (3) can now be expressed in the following dimensionless form

$$\frac{1}{M^2}\frac{d^2u'}{d\eta^2} - \frac{dP}{d\xi} - \theta^\omega(u' - K) = 0 \tag{5a}$$

$$\frac{1}{M^2 mPr}\frac{d^2\theta}{d\eta^2} + \frac{1}{M^2}\left(\frac{du'}{d\eta}\right)^2 + \theta^\omega(u' - K)^2 - u'\left(\frac{dT'}{d\xi} - \frac{dP}{d\xi}\right) = 0 \tag{5b}$$

where η and ξ are the dimensionless coordinates y/h and $x\sigma_c B^2/(\rho u_c)$, respectively; $u' = u/u_c$ is the dimensionless velocity, $M = Bh(\sigma_c/\mu)^{1/2}$ is the Hartmann number; $P = p/(\rho u_c^2)$ is the dimensionless pressure; $K = -E/(u_c B)$ is the loading factor; Pr is the Prandtl number; $m = u_c^2/[c_p(T_c - T_w)]$; and $T' = c_p T_w/u_c^2$ is the dimensionless wall temperature. It will be assumed that σ_c is constant. This implies that $dT_c/dx \ll (T_c - T_w)/h$.

The boundary conditions for Eqs. (5) are that at the wall $u_w' = 0$ and $\theta_w = 0$; at the centerline $du'/d\eta = 0$ and $d\theta/$

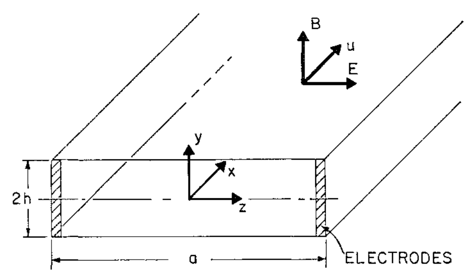

Fig. 1. Schematic of channel showing coordinate system.

$d\eta = 0$; and $dP/d\xi$ and $dT'/d\xi$ must be chosen such that $u_c' = 1$ and $\theta_c = 1$.

Results

When $\omega = 0$ analytic solutions to Eqs. (5a) and (5b) exist. The velocity distribution is the well-known Hartmann profile[8]:

$$u' = [\cosh M - \cosh(M\eta)]/(\cosh M - 1) \qquad (6)$$

where $\eta = 0$ at the centerline. Equation (5b) can now be integrated and an expression for θ results. (Siegel[9] has computed this temperature distribution with the viscous dissipation term in Eq. (3) omitted). The complete solution is:

$$\frac{4F(0)^2}{mPr}\,\theta = M^2(1-\eta^2)\left[2H^2 - 2\left(\frac{dT'}{d\xi} - \frac{dP}{d\xi}\right) \times F(0)\cosh M + 1\right] + F(\eta)\left[4F(0)\left(\frac{dT'}{d\xi} - \frac{dP}{d\xi}\right) - 8H + G(\eta)\right] + [F(\eta)G(\eta) - M^2(1-\eta^2)] \qquad (7)$$

where $F(\eta) = \cosh M - \cosh(M\eta)$, $F(0) = \cosh M - 1$, $G(\eta) = \cosh M + \cosh(M\eta)$, and $H = \cosh M - KF(0)$. The boundary condition $\theta = 1$ at $\eta = 0$ evaluates $(dT'/d\xi - dP/d\xi)$ for any given M, K, mPr. The term $[F(\eta)G(\eta) - M^2(1-\eta^2)]$ would not appear in Eq. (7) if the viscous dissipation had been omitted. It can be shown that when $mPr \sim 0.1$, the contribution of this term is negligible. We will assume that this result also holds for the $\omega \neq 0$ solutions. Values of mPr which can be obtained in an actual channel will be of the order of, or less than this value. For significant magnetic effects we require $M \sim 10$. This sets an upper limit on the velocity since for laminar flow the Reynolds number must satisfy the inequality $Re < 900M$.[10] With this upper limit for u_c and assuming $(T_c - T_w) \sim 1000°C$, $h \sim 1$ cm, and appropriate values for μ, k, and ρ, we obtain $mPr \sim 10^{-1}$.

Solutions for u' and θ when $\omega \neq 0$ were obtained from Eqs. (5a) and (5b), with the term $(du'/d\eta)^2/M^2$ omitted, on an analog computer. For convenience ω was given the values 1 and 2, and the variation in the profiles with ω and K was obtained. The parameter $\bar{m} = \bar{u}^2/[c_p(T_c - T_w)]$ (where— denotes the mean value) rather than m, was held constant at 0.1 so that the Reynolds number is unchanged. The solutions are shown in Fig. 2. Only one θ profile is shown as this was almost independent of ω and K. The velocity distributions can be explained with the momentum equation, which since the flow is fully developed is just a force balance. For the Hartmann solution ($\omega = 0$) for $M \gg 1$, the viscous forces are only important near the walls of the channel, and in the center the Lorentz force $\sigma u_c B^2(u' - K)$ balances the pressure gradient. For $K < 1$ the Lorentz force is a retarding force, and when $\omega \neq 0$ and the electrical conductivity decreases away from the centerline, the velocity must increase to maintain the Lorentz force equal to the accelerating pressure gradient. This explains the overshoots in Fig. 2a. For $K > 1$, an accelerated flow, exactly the opposite occurs and as σ decreases from its centerline value, u' must decrease below its Hartmann solution value to maintain the accelerating Lorentz force equal to the pressure gradient (Fig. 2b). Note that these accelerator results ($K > 1$) are the opposite of those obtained by Hale and Kerrebrock. In their solution, the electrical conductivity increased as one entered the boundary layer, and thus the velocity also increased to maintain the force balance. In this solution the conductivity always decreased away from the centerline. The different conductivity profiles arise because in this solution, with $m \sim 0.1$, the MHD phenomena have little effect on the temperature distribution. The equivalent parameter in the compressible Hale and Kerrebrock solution is M_∞^2 (M_∞ is the freestream Mach number); this is of order 1 or greater, and in this regime the energy equation is substantially affected by the MHD terms.

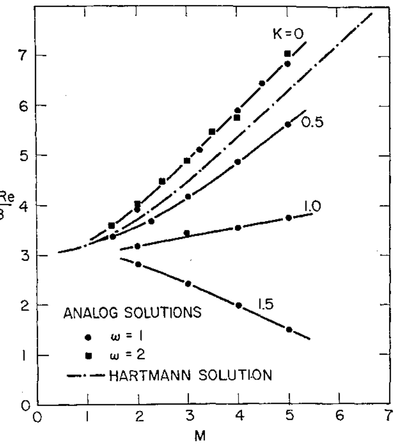

Fig. 3. Friction factor as a function of Hartmann number for $K = 0, 0.5, 1.0$, and 1.5; with $\overline{m}Pr = 0.075$, $\omega = 1$ or 2.

Figure 2b indicates that the wall shear stress changes substantially with K. The friction factor can be evaluated from the velocity profiles as follows:

$$f = 2\tau/(\rho\bar{u}^2) = (u_c/\bar{u})(du'/d\eta)_w 2\mu/(\rho h\bar{u}) \qquad (8)$$

With the Reynolds number defined as $4\rho\bar{u}h/\mu$ we obtain

$$fRe/8 = (du'/d\eta)_w/\bar{u}' \qquad (9)$$

and this product is plotted as a function of M for various values of K in Fig. 3. For $K > 1$, the difference between the $\omega \neq 0$ and the Hartmann solutions is considerable. This difference is due to the different value of the Lorentz force at the wall; for $\omega \neq 0$, $\sigma u_c B^2(u' - K)$ goes to zero at the wall; for $\omega = 0$ this force goes to $-\sigma_c u_c B^2 K$ at the wall. The results indicate that any estimate of f based on the constant fluid properties model may be substantially in error for a gaseous MHD flow.

In conclusion it is pointed out that the aim of this simple model was to illustrate how a variable fluid properties solution can be substantially different from a similar constant fluid properties solution, and to underline the possible error in extending results obtained in an incompressible, constant fluid properties analysis to a real gaseous MHD flow.

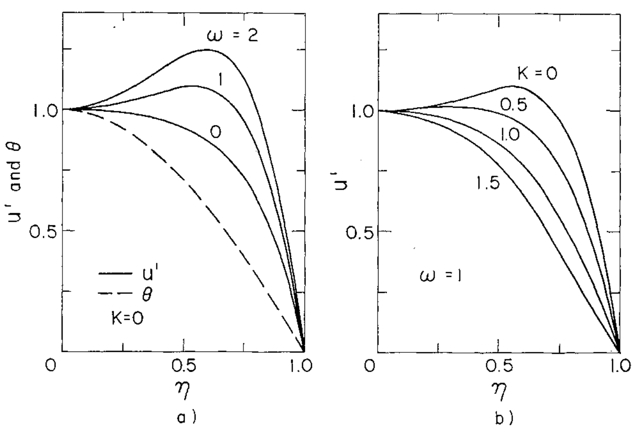

Fig. 2. Velocity and temperature profiles for $M = 5$, $\overline{m}Pr = 0.075$; a) for $\omega = 0, 1$, and 2, with $K = 0$; b) for $K = 0, 0.5, 1.0$, and 1.5, with $\omega = 1$.

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Hypersonic Turbulent Boundary-Layer Interference Heat Transfer in Vicinity of Protuberances

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Introduction

The presence of a protuberance in a local hypersonic flow results in an interaction between the bow shock wave and the local boundary layer. This interaction results in the formation of a lambda-footed bow shock wave and gives rise to severe aerodynamic heating accompanied by an increase in static pressure in a local region in the vicinity of the protuberance. The present paper utilizes a simple hypersonic flow analysis of the interaction between the oblique portion of the lambda-footed bow shock wave and a turbulent boundary layer on a flat plate.

Turbulent Boundary-Layer Lambda-Shock Interaction

A simplified model that retains the essential features of the interaction between the oblique portion of the lambda shock and the turbulent boundary layer is shown in Fig. 1. It is assumed that the upstream foot of the lambda shock can be treated as a two-dimensional oblique shock, since the analysis is concerned with the interaction at the normal portion of the bow shock.

Experiments described in Ref. 1 revealed that the high pressure downstream of the normal portion of the bow shock feeds upstream in the subsonic part of the boundary layer resulting in a thickening of the boundary layer. This rather abrupt growth of the boundary layer is idealized in Fig. 1 as the formation of an effective "wedge" of angle θ with an "attached" oblique shock such that $p_0 < p < p_{\text{stag}}$.

It is clear that the essential features of the present simplified model for the turbulent boundary-layer lambda-shock interaction are similar to those used in the laminar "weak interaction" problem[2] with the exception, of course, that the turbulent skin-friction law will be used in the present case.

It is well known that the relation between skin-friction coefficient and Reynolds number is nearly independent of Mach number when the fluid properties are evaluated at the wall temperature. Thus, the incompressible form of the skin-friction law expressed as

$$c_{f_w} = C(\text{Rey}_w)^m \tag{1}$$

may be assumed to hold for high speeds, where C and m are constants. Now the skin-friction coefficient c_{f_w} is defined as

$$c_{fw} \equiv 2\tau_w/\rho_w V^2 = (2\tau_w/\rho_0 V_0^2)(V_0/V)^2(\rho_0/\rho_w) = c_f(V_0/V)^2(\rho_0/\rho_w) \tag{2}$$

where τ_w is the wall shear stress, c_f is the local compressible skin-friction coefficient based on the freestream dynamic pressure $\rho_0 V_0^2$, and V is the resultant velocity downstream of the oblique portion of the lambda shock corresponding to the pressure p.

Since, for hypersonic speeds, $V/V_0 \approx 1$, then using the equation of state, Eqs. (1) and (2) may be combined to yield

$$c_f = C(p_w/p_0)(T_0/T_w)(\text{Rey}_w)^m \tag{3}$$

Assuming an n-power law for the viscosity-temperature relation and $V/V_0 \approx 1$, it follows that the wall Reynolds number at station B, Rey_w, and the freestream Reynolds number, Rey_0, are related as

$$\text{Rey}_w = (p_w/p_0)(T_0/T_w)^{1+n}\,\text{Rey}_0 \tag{4}$$

Substituting Eq. (4) into Eq. (3) and recognizing that $\partial p/\partial y = 0$ in the boundary layer gives

$$c_f/(\text{Rey}_0)^m = C(p/p_0)^{1+m}(T_0/T_w)^{1+m(1+n)} \tag{5}$$

For hypersonic speeds, the temperature-Mach number relation, assuming a recovery factor of unity, may be approximated as

$$T_w/T_0 = 1 + [(\gamma - 1)/2]M_0^2 \approx [(\gamma - 1)/2]M_0^2 = \xi_0 \tag{6}$$

so that Eq. (5) becomes

$$c_f/(\text{Rey}_0)^m = [C/(\xi_0)^{1+m\,(1+n)}](p/p_0)^{1+m} \tag{7}$$

This is the expression for local skin-friction coefficient c_f at station B that takes into account the interaction between the oblique portion of the lambda shock and the turbulent boundary layer.

In the absence of the protuberance, i.e., the *noninteraction case*, $p/p_0 = 1$, and Eq. (7) reduces to

$$c_{f_0}/(\text{Rey}_0)^m = C/(\xi_0)^{1+m\,(1+n)} \tag{8}$$

where c_{f_0} is referred to as the turbulent skin-friction coefficient at station B without interaction. It follows, therefore, that the effect of the turbulent boundary-layer lambda-shock

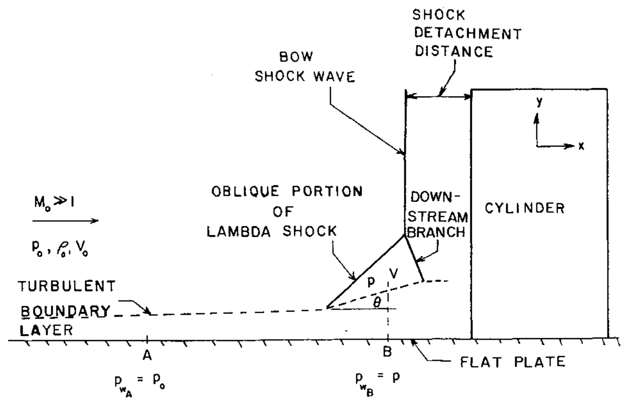

Fig. 1 Interaction of lambda shock and turbulent boundary layer (schematic).

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